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Non-integrable Hamiltonian systems and the Heun equation

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We study a family of Hamiltonian systems which is a perturbation of the Calogero–Moser system. The presence of the coupling terms in the Hamiltonian makes the system non-integrable. We prove this non-integrability by using the monodromy group of the linearization of the complex flow around particular solutions and the stability of the associated Heun equation.

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1. Introduction

In this article we study the family of Hamiltonian systems with two degrees of freedom defined by

$$H(x_1, x_2, y_1, y_2) = \tfrac{1}{2}(y_1^2 + y_2^2) + 1/(x_1 - x_2)^2 + K(x_1, x_2)\,, \tag{1}$$

with

$$K(x_1, x_2) = a(x_1 + x_2)^2 + b(x_1^4 + x_2^4) + cx_1^2x_2^2 + dx_1x_2(x_1^2 + x_2^2)\,,$$

and the symplectic form

$$\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2\,.$$

This family depends on four real parameters a, b, c, d. When these parameters vanish altogether, (1) defines the Calogero–Moser system with two degrees of freedom, which is integrable [1].

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The methods we use are not perturbative in character; we prove the non-integrability of (1) for an explicit full open set of values of a, b, c, d in the parameter space, which is not only a neighbourhood of the Calogero–Moser system. A comparison of this study with that in ref. [2] shows that the non-integrability is connected with the presence of the coupling terms in the Hamiltonian. On the other hand, numerical computations [3] reveal a chaotic behaviour for a potential of sixth order without coupling term.

Our results have been announced in ref. [4].

Theorem 1.1.

(i) If $2b+c-2d\neq 0$,

(ii) if

(iia) $(c-6b)/(2b+c-2d)>0$ *and* $a\leqslant 0$, *or*

(iib) $(c-6b)/(2b+c-2d)>0$ *and* $0<a<\frac{1}{4}(c-6b)/(2b+c-2d)^{1/3}$, *or*

(iic) $(c-6b)/(2b+c-2d)<0$ *and* $a<-\frac{1}{4}|(c-6b)/(2b+c-2d)^{1/3}|$, *and*

(iii) if $(50b-7c-2d)/(2b+c-2d)\neq(2p+1)^2$, $\forall p\in\mathbb{N}$,

then the Hamiltonian system defined by (1) does not possess a second integral which is meromorphic and functionally independent of H in the set:

(iv) $1<H/2h^*$ *in case* (*iia*), *or*

$$1<H/2h^*<\frac{(16a^2/3)(2b+c-2d)^{2/3}}{(c-6b)^2}+\frac{(1/6a)(c-6b)}{(2b+c-2d)^{1/3}},$$

in cases (iib) and (iic), where $h^*=\frac{3}{8}(2b+c-2d)^{1/3}$.

The proof is inspired by a theorem of Ziglin [5], which gives restrictive conditions on the linearized equations near a particular solution of a Hamiltonian system which possesses a meromorphic supplementary integral: Suppose that the Hamiltonian system has a family of particular solutions Γ_h which are parametrized by elliptic functions of complex time. Let us assume that this family Γ_h depends analytically on the parameter $h\in\,]h_0, h_1[$, and that the projection $(x, y)\mapsto x$ restricted to Γ_h defines a covering of Γ_h on $\mathbb{C}$. Let G be the monodromy group of the normal variational equation associated to the solution Γ_h. We say that an element g of G is non-resonant if none of its eigenvalues is a root of unity. Ziglin proved [5] that, if the Hamiltonian system has an integral F which is meromorphic and functionally independent of H in a neighbourhood of Γ_h, and if G contains a non-resonant element, then G is solvable (i.e., it contains an Abelian subgroup of finite index).

2. Determination of the elliptic solutions Γ_h

The Hamiltonian system generated by H is

$$\dot{x}_i = \partial H/\partial y_i\,, \qquad \dot{y}_i = -\partial H/\partial x_i\,, \quad i=1,2\,. \tag{2}$$

This system admits the particular solutions Γ_h defined by

$$x_1 = -x_2 = x\,,$$

with

$$\ddot{x} = 1/4x^3 - 2(2b+c-2d)x^3\,.$$

This differential equation has the first integral

$$h = H/2 = \tfrac{1}{2}\dot{x}^2 + 1/8x^2 + (b+\tfrac{1}{2}c-d)x^4\,,$$

which can be written, by defining $u=x^2$, as

$$\dot{u}^2 = -4(2b+c-2d)u^3 + 8hu - 1\,. \tag{3}$$

If the coefficient $(2b+c-2d)$ does not vanish, the linear change $v=-(2b+c-2d)u$ transforms (3) into the normal form

$$\dot{v}^2 = 4v^3 - \gamma_2 v - \gamma_3\,, \tag{4}$$

the solutions of which are the Weierstrass elliptic functions $\wp\,(t, \gamma_2, \gamma_3)$, with $t \in \mathbb{C}$. The invariants γ_2 and γ_3 are defined by

$$\gamma_2 = 8h(2b+c-2d)\,,$$
$$\gamma_3 = (2b+c-2d)^2\,,$$

and they determine the two complex periods of $\wp$ [6]. As γ_2 depends on h linearly, the Weierstrass function $\wp$ depends on h analytically.

It is possible to express the Weierstrass functions $\wp$ in terms of the Jacobi elliptic function sn [6,7], if the polynomial

$$P(v) = 4v^3 - \gamma_2 v - \gamma_3 \tag{5}$$

has its roots α_i all real and distinct. This occurs when the discriminant of eq. (5) is negative, i.e.,

$$h/h^* > 1\,, \quad h^* = \tfrac{3}{8}(2b+c-2d)^{1/3}\,.$$

Then we have $\alpha_1+\alpha_2+\alpha_3=0$ and $\alpha_1\alpha_2\alpha_3=\gamma_3/4>0$, so that $\alpha_1<\alpha_2<0<\alpha_3$.

Following ref. [7], let us write

$$v = \alpha_1 + (\alpha_2 - \alpha_1)w^2\,, \tag{6}$$

$$\mathrm{d}z = \sqrt{\alpha_3 - \alpha_1}\;\mathrm{d}t \tag{7}$$

with $k^2 = (\alpha_2-\alpha_1)/(\alpha_3-\alpha_1) \in \,]0, 1[$. The differential equation (4) becomes

$$(\mathrm{d}w/\mathrm{d}z)^2 = (1-w^2)(1-k^2w^2)\,. \tag{8}$$

This is the Legendre normal form of the Jacobi equation [8], the solution of which is the Jacobi elliptic function $w(z)=\mathrm{sn}(z,k)$. Thus, the solution of eq. (4), which characterizes the doubly periodic orbits Γ_h, can be written as

$$v(t)=\alpha_1+(\alpha_2-\alpha_1)\mathrm{sn}^2(\sqrt{\alpha_3-\alpha_1}\,t,k)\ . \tag{9}$$

Recall [6] that the elliptic function sn has two periods, $\omega_1\in\mathbb{R}$ and ω_2 purely imaginary. ω_1 and $\omega_2/\sqrt{-1}$ are given by elliptic integrals of the first kind. But the function sn is also anti-periodic (period $\omega_1/2$) so that sn^2 has two periods $\omega_1/2\in\mathbb{R}$ and $\omega_2\in\mathbb{C}$. On the other hand, sn has two simple poles in each parallelogram of periods; then sn^2 has only one pole z_∞ of order two in its parallelogram of periods $(0,\omega_1/2,\omega_1/2+\omega_2,\omega_2)$. If t is real, $\mathrm{sn}^2 t$ *is* $\omega_1/2$ periodic and bounded, because the pole z_∞ is not real. The differential equation (8) shows that the upper bound of sn^2 is 1, and therefore

$$\mathrm{sn}^2(\sqrt{\alpha_3-\alpha_1}\,t,k)\in[0,1]\ ,\quad \forall t\in\mathbb{R}\ . \tag{10}$$

3. Normal variational equation associated with the solutions Γ_h

The differential equations which describe the solutions near Γ_h are given by the variational system deduced from (2) along Γ_h,

$$(\mathrm{d}/\mathrm{d}t)\begin{pmatrix}\xi_1\\ \xi_2\\ \zeta_1\\ \zeta_2\end{pmatrix}=JM\begin{pmatrix}\xi_1\\ \xi_2\\ \zeta_1\\ \zeta_2\end{pmatrix},$$

where

$$J=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$$

and M is the Hessian matrix of the Hamiltonian function H, computed along the solutions Γ_h. This variational system can be written as

$$(\mathrm{d}^2/\mathrm{d}t^2)\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\begin{bmatrix}-A-B & A-C\\ A-C & -A-B\end{bmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix},$$

with

$$A=3/8u^2,\qquad B=2a+2(6b+c-3d)u\ ,\qquad C=2a-2(2c-3d)u\ ,$$

where $u(t)$ is the family of elliptic functions defined by (3). This system can be decoupled through the symplectic transformation $(\xi_1,\xi_2,\zeta_1,\zeta_2)\mapsto(\eta_1,\eta_2,\chi_1,\chi_2)$ defined by

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \chi_1\\ \chi_2\end{pmatrix} = (\sqrt{2}/2)\begin{pmatrix}1 & -1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 1 & 1\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \zeta_1\\ \zeta_2\end{pmatrix}.$$

Then the variational system decouples into the form

$$\ddot{\eta}_1 = (-2A-B+C)\eta_1\,,$$
$$\ddot{\eta}_2 = -(B+C)\eta_2\,.$$

The second equation of this system is the normal variational equation [5]. If we denote η_2 by η, this equation can be written as follows:

$$\ddot{\eta}+C(t,h)\eta=0\,, \tag{11}$$

with

$$C(t,h)=4a+[2(c-6b)/(2b+c-2d)]$$
$$\times[\alpha_1+(\alpha_2-\alpha_1)\mathrm{sn}^2(\sqrt{\alpha_3-\alpha_1}\,t,k)]\,. \tag{11'}$$

We have to study the monodromy group associated with eq. (11), which is linear, homogeneous, of second order, and where the coefficient $C(t,h)$ is a doubly periodic function of complex time t. Moreover, if t is real, $C(t,h)$ is bounded, as we showed previously.

The normal variational equation (11) is a Heun equation which, in some particular cases, becomes a Lamé equation [9]:

Proposition 3.1. *Equation (11) is a Lamé equation,*

$$\ddot{\eta}=(p(p+1)\wp(t)-4a)\eta\,,$$

if and only if

$$\frac{50b-7c-2d}{2b+c-2d}=(2p+1)^2\,,\quad p\in\mathbb{N}\,.$$

Proof. Equation (11) can also be written as follows:

$$\ddot{\eta}+\left(4a-\frac{2(6b-c)}{2b+c-2d}v\right)\eta=0\,,$$

where v is the solution of (4), i.e., $v=\wp(t,\gamma_2,\gamma_3)$. This equation is a Lamé's equation if and only if [6]

$$\frac{2(6b-c)}{2b+c-2d}=p(p+1)\,,\quad p\in\mathbb{N}\,,$$

i.e.,

$$\frac{50b-7c-2d}{2b+c-2d}=(2p+1)^2 .$$

Condition (iii) is put precisely to exclude this case. □

4. The monodromy group G of the normal variational equation

Since the coefficient $C(t, h)$ has only one pole $t_\infty=z_\infty/\sqrt{\alpha_3-\alpha_1}$ of order two in each parallelogram of periods, eq. (11) is of the Fuchs type, i.e., the pole t_∞ is regular singular. Then we know [9–11] that there exist two solutions η_a and η_b which form a fundamental system and have an expansion of the following form:

$$\eta_{a,b}=\tau^\nu(1+c_1\tau+c_2\tau^2+\cdots) . \qquad (12)$$

These expansions converge in a neighbourhood of $\tau=t-t_\infty=0$.

Following the Frobenius method [9], we can write down the characteristic equation,

$$\nu(\nu-1)+2(c-6b)/(2b+c-2d)=0 ,$$

and compute the exponents ν_1 and ν_2 corresponding to the fundamental solutions η_a and η_b,

$$\nu_{1,2}=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{(50b-7c-2d)/(2b+c-2d)} . \qquad (13)$$

These exponents do not depend on the energy h, although the pole t_∞ does.

In order to define the monodromy group of the normal variational equation, we must follow the fundamental solutions η_a and η_b along a path in the complex plane (t) from t to t' such that $t'-t=m\omega_1/2+n\omega_2$, with m and $n\in\mathbb{Z}$. The solutions η_a and η_b become $\tilde{\eta}_a$ and $\tilde{\eta}_b$, which also define a fundamental system,

$$\begin{pmatrix}\tilde{\eta}_a\\ \tilde{\eta}_b\end{pmatrix}=M\begin{pmatrix}\eta_a\\ \eta_b\end{pmatrix} .$$

As eq. (11) is a Hamiltonian system, the matrices M are symplectic, $M\in SL(2, \mathbb{C})$. They generate the monodromy group G of the Heun equation. We can find two particular generators g_1 and g_2, which we choose as follows: g_1 and g_2 correspond to the real and the complex periods $\omega_1/2$ and ω_2, respectively, of the coefficient $C(t, h)$.

We can now compute the commutator $g^*=g_2^{-1}g_1^{-1}g_2g_1$ and prove

Proposition 4.1. *The commutator g^* is different from the identity if and only if condition (iii) of theorem 1.1 is fulfilled.*

Proof. Each period parallelogram of the coefficient $C(t, h)$ of eq. (11) contains only one double pole. Thus, the commutator g^* is the matrix corresponding to an anti-clockwise loop around this pole t_∞. The expansions (12) show how the fundamental solutions are transformed along this loop: η_a and η_b are multiplied by $\exp(2\pi i \nu_j)$, $j=1, 2$. Therefore, g^* is a diagonal matrix and its eigenvalues Λ_j are equal to 1 if and only if the exponents ν_j defined in (13) are integers, i.e., if and only if condition (iii) is not satisfied. □

In order to prove that the monodromy matrix g_1 is non-resonant, we study the instability of solutions of eq. (11), and we use the following

Lemma 4.2. *If $\Gamma(t)<0$ for any real non-negative t, the matrix $R(t)$ of the fundamental solutions of the linear equation $\ddot{y}+\Gamma(t)y=0$ has the following property: $\operatorname{tr} R(t)>2$, for any strictly positive t.*

Proof. If η_α and η_β are two fundamental solutions of the differential equation such that $\eta_\alpha(0)=\dot{\eta}_\beta(0)=1$ and $\dot{\eta}_\alpha(0)=\eta_\beta(0)=0$, the matrix $R(t)$ is defined by

$$R(t)=\begin{bmatrix}\eta_\alpha(t) & \eta_\beta(t)\\ \dot{\eta}_\alpha(t) & \dot{\eta}_\beta(t)\end{bmatrix}.$$

If $\Gamma(t)<0$ for any real non-negative t, it can be proved, as in ref. [12], that $\eta_\alpha(t)$ and $\dot{\eta}_\beta(t)$ are strictly increasing functions, and

$$\operatorname{tr} R(t)=\eta_\alpha(t)+\dot{\eta}_\beta(t)>2\,, \quad \text{for any } t>0\,. \qquad \square$$

Theorem 4.3. *If conditions (i), (ii) and (iv) of theorem 1.1 are satisfied, the coefficient $C(t, h)$ of eq. (11) is strictly negative for any real value of t.*

Proof. We want to find sufficient conditions such that $C(t, h)<0$ for every real t. Condition (10) shows that

$$\alpha_1 \leqslant \alpha_1+(\alpha_2-\alpha_1)\operatorname{sn}^2(\sqrt{\alpha_3-\alpha_1}\,t, k)\leqslant \alpha_2\,,$$

for any real value of t, and therefore

(α) if

$$(c-6b)/(2b+c-2d)>0\,,$$

then

$$C(t, h)\leqslant 4a+(2(c-6b)/(2b+c-2d))\alpha_2\,,$$

(β) if

$$(c-6b)/(2b+c-2d)<0\,,$$

then

$$C(t,h) \leqslant 4a + (2(c-6b)/(2b+c-2d))\alpha_1 ,$$

where α_1, α_2, α_3 are the three roots of $P(v)$ defined by (5). These roots are real if $h/h^* > 1$ and then we have $\alpha_1 < \alpha_2 < 0 < \alpha_3$. (We do not consider the case $c-6b=0$, because it does not satisfy condition (iii) of non-integrability.)

We want that the upper bound of $C(t, h)$, in both cases (α) and (β), be negative, and thus we must find a majoration for the numbers α_2 (in case α) or $-\alpha_1$ (in case β). This can be done more easily if we transform eq. (5) by setting

$$v = (8h^*/3)^2 V ,$$

where h^* is defined in condition (iv). Equation (5) then becomes

$$Q(V) = 0 , \tag{14}$$

with

$$Q(V) = 4V^3 - (3h/h^*)V - 1 . \tag{15}$$

Equation (14) admits three real roots β_i, which depend on h/h^* only, and satisfy

$$\alpha_i = (8h^*/3)^2 \beta_i , \qquad \sum_1^3 \beta_i = 0 , \qquad \beta_1 < \beta_2 < 0 < \beta_3 .$$

In case (α), i.e., if $(c-6b)/(2b+c-2d) > 0$, we want

$$4a + (2(c-6b)/(2b+c-2d))\alpha_2 < 0 ,$$

i.e.,

$$\beta_2 < \bar{V} , \quad \text{with } \bar{V} = 2a \frac{(2b+c-2d)^{1/3}}{6b-c} . \tag{16}$$

As β_2 is the largest negative root of eq. (14), condition (16) is satisfied in two cases:

- if $\bar{V} \geqslant 0$, i.e., if $a \leqslant 0$, without any supplementary condition on h;
- if $\bar{V} < 0$, i.e., if $a > 0$; then we must have

$$Q(\bar{V}) < 0 \quad \text{and } (\mathrm{d}Q/\mathrm{d}V)(\bar{V}) < 0 ,$$

i.e.,

$$\varphi_2(\bar{V}) < h/h^* < \varphi_1(\bar{V}) , \quad \text{with } \varphi_1(V) = \tfrac{4}{3}V^2 - 1/3V , \quad \varphi_2(V) = 4V^2 .$$

The set of values of h defined by these conditions is not empty if and only if

$$a < \tfrac{1}{4}(c-6b)/(2b+c-2d)^{1/3} .$$

Then we have $\varphi_2(\bar{V}) < 1$, so that the interval for h/h^* is characterized by condition (iv) in case (iib).

This ends the proof for case (α), i.e., when conditions (iia) or (iib) are satisfied. *In case (β)*, i.e., if $(c-6b)/(2b+c-2d)<0$, we want

$$4a+\frac{2(c-6b)}{2b+c-2d}\alpha_1<0\,,$$

i.e.,

$$\beta_1>\bar{V}\,,\quad \text{with } \bar{V}=2a\frac{(2b+c-2d)^{1/3}}{6b-c}\,. \tag{17}$$

As β_1 is the smallest root of eq. (14), condition (17) is equivalent to

$$\bar{V}<0 \quad \text{and} \quad Q(\bar{V})<0 \quad \text{and} \quad (\mathrm{d}Q/\mathrm{d}V)(\bar{V})>0\,,$$

i.e.,

$$h/h^*<\varphi_1(\bar{V}) \quad \text{and} \quad h/h^*<\varphi_2(\bar{V})\,.$$

The set of values of h defined by these conditions is not empty if

$$a<-\tfrac{1}{4}\,|\,(c-6b)/(2b+c-2d)^{1/3}|\,.$$

Furthermore, $\bar{V}$ always satisfies $\bar{V}<-1/2$, and thus $\varphi_1(\bar{V})<\varphi_2(\bar{V})$. Therefore, the previous conditions can be replaced by the first one only. This completes the proof of theorem 4.3. □

Proposition 4.4. *If conditions (i), (ii) and (iv) of theorem 1.1 are satisfied, the monodromy matrix g_1 is non-resonant.*

Proof. The previous theorem shows that we can apply lemma 4.2 to the variational equation (11). Thus the fundamental matrix $R(t)$ associated is such that $\mathrm{tr}\, R(t)>2$ for any strictly positive t. But the monodromy matrix g_1 is defined as $g_1=R(\omega_1/2)$. Thus, $\mathrm{tr}\, g_1>2$, and the eigenvalues λ_j of g_1 cannot be roots of unity, because $\lambda_1\lambda_2=1$. Therefore, g_1 is non-resonant. Furthermore, λ_1 and λ_2 are real numbers, because $g_1\in \mathrm{SL}(2,\mathbb{R})$. □

Proposition 4.5. *If conditions (i), (ii) and (iv) of theorem 1.1 are satisfied, the monodromy matrices g_1 and g_2 are such that $g^*\neq g_1^2$.*

Proof. We know that the eigenvalues Λ_j of g^* are explicitly given by

$$\Lambda_j=\exp(2\pi\mathrm{i}\,\nu_j)\,,\quad j=1,2\,,$$

where the ν_j are defined in (13). These eigenvalues satisfy $\Lambda_1+\Lambda_2=\mathrm{tr}\, g^*\in\mathbb{R}$ and $\mathrm{tr}\, g^*\leqslant 2$.

On the other hand, we proved that tr $g_1 > 2$ if conditions (i), (ii) and (iv) are satisfied. But $\mathrm{tr}(g_1^2) = (\mathrm{tr}\, g_1)^2 - 2$, because g_1 is real and symplectic. This implies $\mathrm{tr}(g_1^2) > 2$, and that the matrices g^* and g_1^2 cannot be equal. □

According to Ziglin's theorem [5], we have obtained an obstruction to the integrability of the system, and this ends the proof of theorem 1.1.

5. Some integrable cases

When condition (iii) of theorem 1.1 is not satisfied, we find several cases of integrability of the system.

(a) If $c=6b$, the symplectic transformation defined by

$$q_1 = (x_1 + x_2)/\sqrt{2}, \qquad q_2 = (x_1 - x_2)/\sqrt{2},$$
$$p_1 = (y_1 + y_2)/\sqrt{2}, \qquad p_2 = (y_1 - y_2)/\sqrt{2},$$

gives

$$H(q, p) = (p_1^2 + p_2^2)/2 + 2aq_1^2 + (2b + d/2)q_1^4 + 1/2q_2^2 + (2b - d/2)q_2^4 .$$

As this Hamiltonian is separable in an obvious way, the system is integrable.

(b) The case defined by $c=2b$ and $a=d=0$ is separable. Indeed, the Hamiltonian can be written in polar coordinates as

$$H(\rho, \theta, p_\rho, p_\theta) = (p_\rho^2 + p_\theta^2/\rho^2)/2 + 1/\rho^2(\cos\theta - \sin\theta)^2 + b\rho^4 .$$

If we define the function K by $H=K/\rho^2$, the Hamiltonian system generated by H takes the form

$$\begin{aligned}\mathrm{d}\rho/(1/\rho^2)(\partial K/\partial p_\rho) &= \mathrm{d}\theta/(1/\rho^2)(\partial K/\partial p_\theta)\\ &= \mathrm{d}p_\rho/[(-1/\rho^2)(\partial K/\partial \rho) + (K/\rho^4)(\partial\rho^2/\partial\rho)]\\ &= \mathrm{d}p_\theta/(-1/\rho^2)(\partial K/\partial\theta) = \mathrm{d}t .\end{aligned}$$

If we change the independent variable by putting $\mathrm{d}t=\rho^2\,\mathrm{d}\tau$, and if we remark that, along any solution of energy h, we have $K/\rho^2 = H = h$, this Hamiltonian system becomes

$$\begin{aligned}\mathrm{d}\rho/(\partial K/\partial p_\rho) &= \mathrm{d}\theta/(\partial K/\partial p_\theta)\\ &= \mathrm{d}p_\rho/((-\partial K/\partial\rho) + h\,\partial\rho^2/\partial\rho)\\ &= \mathrm{d}p_\theta/(-\partial K/\partial\theta) = \mathrm{d}\tau .\end{aligned}$$

This is exactly the Hamiltonian system generated by the Hamiltonian

$\mathscr{H}=K-hp^2=\rho^2(H-h)$. In this system, we consider only the solutions which satisfy the invariant relation $\mathscr{H}=0$, i.e., $H=h$ (cf. ref. [13]).

Now, the Hamiltonian $\mathscr{H}$ has the following form:

$$\mathscr{H}=(\rho^2 p_\rho^2+p_\theta^2)/2+1/(\cos\theta-\sin\theta)^2+b\rho^6-h\rho^2\,,$$

and it is separable.

Some particular integrable cases of the system we consider were introduced and investigated first by Inozemtsev [14].

6. Non-integrability of the system in the limit $h\to+\infty$

In this section, we consider the previous particular family of solutions Γ_h defined by the differential equation

$$\ddot{x}=1/4x^3-2(2b+c-2d)x^3\,, \tag{18}$$

or by its first integral

$$h=H/2=\dot{x}^2/2+1/8x^2+(b+c/2-d)x^4\,, \tag{19}$$

and the normal variational equation along Γ_h,

$$\ddot{\eta}+[4a+2(6b-c)x^2]\eta=0\,. \tag{20}$$

The scaling

$$x=h^{1/4}(b+c/2-d)^{-1/4}\varphi\,,$$

$$\eta=h^{1/4}(b+c/2-d)^{-1/4}\xi\,,$$

$$t=h^{-1/4}\cdot\tfrac{1}{2}(b+c/2-d)^{-1/4}\tau\,, \tag{21}$$

transforms equations (18), (19), (20) into the following form, in the limit $h\to+\infty$:

$$\mathrm{d}^2\varphi/\mathrm{d}\tau^2+\varphi^3=0\,,$$

$$2(\mathrm{d}\varphi/\mathrm{d}\tau)^2=1-\varphi^4\,,$$

$$\mathrm{d}^2\xi/\mathrm{d}\tau^2+\frac{6b-c}{2b+c-2d}\varphi^2\xi=0\,. \tag{22}$$

This is exactly the system associated to a homogeneous potential of degree $k=4$ [15]. The normal variational equation of the homogeneous problem can be transformed into a Gauss hypergeometric equation. The study of its monodromy group was performed by Yoshida, who gave sufficient conditions for non-integrability of the system [15]. By the use of an "integrability coefficient" λ_{I} as the intrinsic parameter of the normal variational equation

$$d^2\xi/d\tau^2+\lambda_1\varphi^{k-2}\xi=0\ ,$$

Yoshida determines the following non-integrability region S_k for this coefficient (with $k=4$ in our problem):

$$S_4=]-\infty,0[\cup]1,3[\cup]6,10[\cup\cdots\cup]j(2j-1),j(2j+1)[\cup\cdots.$$

The integrability coefficient of our system is defined by

$$\lambda_1=(6b-c)/(2b+c-2d)\ .$$

Thus, the application of Yoshida's criterion gives

Proposition 6.1. *If the coefficient* $(6b-c)/(2b+c-2d)$ *is in the above region* S_4, *the Hamiltonian system (1) is non-integrable in the sense of Ziglin, in the limit* $h\to+\infty$.

Remark. Condition (iii) of theorem 1.1 is not fulfilled, i.e.,

$$\frac{50b-7c-2d}{2b+c-2d}=(2p+1)^2\ ,\quad p\in\mathbb{N}$$

(cf. theorem 1.1) if and only if

$$\frac{6b-c}{2b+c-2d}\in\{0,1,3,6,10,\ldots,p(p+1)/2,\ldots\}\ .$$

These integers are the boundaries of the intervals which appear in the non-integrability region S_4 of the coefficient $(6b-c)/(2b+c-2d)$.

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